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A New Proof of A. F. Timan's Approximation Theorem, II

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1. INTRODUCTION

The object of this paper is to give a simple new proof of Timan's [7] approximation theorem based on the values of the given function at the zeros of Tchebycheff polynomials. The main idea of this paper is closely related to the extremely interesting work of Bernstein [2], concerning a modification of the formula of Lagrange, and to our earlier result [5]. Let

$$T_n(x) = \cos n\theta, \qquad \cos\theta = x \tag{1.1}$$

be the Tchebycheff polynomial of degree n. T_n has all its zeros in $(-1, 1)$, at the points

$$x_{kn} = \cos((2k-1)/2n)\pi = \cos\theta_{kn}, \qquad k = 1, 2,..., n. \tag{1.2}$$

We denote by

$$\ell_{kn}(x) = \frac{(-1)^{k+1}(1-x_{kn}^2)^{1/2}}{n}\,\frac{T_n(x)}{x - x_{kn}} \tag{1.3}$$

the fundamental polynomials of Lagrange interpolation based on the nodes x_{kn}. Bernstein [2] considered the following interpolation process based on the nodes (1.2).

$$R_n[f, x] = \sum_{k=1}^{n} f(x_{kn})\,\varphi_{kn}(x), \tag{1.4}$$

where

$$\begin{aligned} \varphi_{1n}(x) &= \frac{3\ell_{1n}(x) + \ell_{2n}(x)}{4}, \qquad \varphi_{n-1,n}(x) = \frac{\ell_{n-1,n}(x) + 3\ell_{nn}(x)}{4}, \\ \varphi_{kn}(x) &= \frac{\ell_{k-1,n}(x) + 2\ell_{kn}(x) + \ell_{k+1,n}(x)}{4}, \qquad k = 2, 3,..., n-1. \end{aligned} \tag{1.5}$$

Concerning $R_n[f, x]$ he proved the following.

THEOREM 1.1. *Let $f(x)$ be an arbitrary continuous function in* $[-1, +1]$; *then $R_n[f, x]$ as defined by* (1.4) *converges uniformly to $f(x)$ on* $[-1, +1]$.

We aim to give a precise estimate of $R_n[f, x] - f(x)$ in the following form.

THEOREM 1.2. *Let $f(x)$ be an arbitrary continuous function in* $[-1, +1]$ *having $\omega(\delta)$ as its modulus of continuity. Then there exists a positive constant c independent of n, x, and f such that*

$$| R_n[f, x] - f(x)| \leqslant c \left[\omega\left(\frac{(1-x^2)^{1/2}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right]. \tag{1.6}$$

Thus $R_n[f, x]$ as defined by (1.4) provides a new proof of Timan's approximation theorem. Compare our proof with order new proofs (see [3, 5, 6, and 8]).

2. PRELIMINARIES

For the sake of brevity let us fix n and let us denote x_{kn} by x_k, $\ell_{kn}(x)$ by $\ell k(x)$ and $\varphi_{kn}(x)$ by $\varphi_k(x)$. From

$$\sin\frac{\theta_k - \theta}{2} = \sin\frac{\theta_k}{2}\cos\frac{\theta}{2} - \cos\frac{\theta_k}{2}\sin\frac{\theta}{2}$$

it follows that

$$\left|\sin\frac{\theta_k - \theta}{2}\right| \leqslant \sin\frac{\theta_k}{2}\cos\frac{\theta}{2} + \cos\frac{\theta_k}{2}\sin\frac{\theta}{2} = \sin\frac{\theta + \theta_k}{2}. \tag{2.1}$$

Since we can write

$$\frac{2\sin\theta_k}{\cos\theta - \cos\theta_k} = \cot\frac{\theta_k - \theta}{2} + \cot\frac{\theta + \theta_k}{2},$$

we obtain from (1.3)

$$\ell_k(x) = \frac{(-1)^k \cos n\theta}{2n}\left[\cot\frac{\theta_k - \theta}{2} + \cot\frac{\theta_k + \theta}{2}\right]. \tag{2.2}$$

From (1.5) and (2.2) we have another representation of ($k = 2, 3,..., n-1$)

$$\begin{aligned}\varphi_k(x) = \frac{(-1)^k \cos n\theta}{2n}\Big[&\cot\frac{\theta_{k-1} - \theta}{2} - 2\cot\frac{\theta_k - \theta}{2} + \cot\frac{\theta_{k+1} - \theta}{2} \\ &+ \cot\frac{\theta_{k-1} + \theta}{2} - 2\cot\frac{\theta_k + \theta}{2} + \cot\frac{\theta_{k+1} + \theta}{2}\Big].\end{aligned} \tag{2.3}$$

On using the trigonometric formula

$$\cot(\alpha + h) - 2\cot\alpha + \cot(\alpha - h) = \frac{2\sin^2 h \cot\alpha}{\sin(\alpha - h)\sin(\alpha + h)} \tag{2.4}$$

for $\alpha = (\theta_k \pm \theta)/2$ and $h = \pi/2n$ (2.3) can be written as

$$\varphi_k(x) = \frac{(-1)^k \cos n\theta \sin^2(\pi/2n)}{n}\left[\frac{\cot[(\theta_k - \theta)/2]}{\sin[(\theta_{k-1} - \theta)/2]\sin[(\theta_{k+1} - \theta)/2]}\right.$$
$$\left. + \frac{\cot[(\theta_k + \theta/2)}{\sin[(\theta_{k-1} + \theta)/2]\sin[(\theta_{k+1} + \theta)/2]}\right]. \tag{2.5}$$

A direct computation shows that

$$\varphi_1(x) = \frac{\cos n\theta \sin^3(\pi/2n)(\cos\theta + 2\cos\theta_1)}{n(\cos\theta - \cos\theta_1)(\cos\theta - \cos\theta_2)}, \tag{2.6}$$

and

$$\varphi_n(x) = \frac{(-1)^{n+1}\cos n\theta \sin^3(\pi/2n)(\cos\theta + 2\cos\theta_n)}{n(\cos\theta - \cos\theta_{n-1})(\cos\theta - \cos\theta_n)}. \tag{2.7}$$

Finally we need ($k = 2, 3,\ldots, n-3, n-2$)

$$\varphi_k(x) + \varphi_{k+1}(x) = \frac{(-1)^k \cos n\theta \sin^3(\pi/2n)}{n}[I_k(\theta) + I_k(-\theta)], \tag{2.8}$$

where

$$I_k(\theta) = \frac{2\cos(\pi/2n) + \cos(\theta_k + \theta + (\pi/2n))}{\sin[(\theta_{k-1} + \theta)/2]\sin[(\theta_k + \theta)/2]\sin[(\theta_{k+1} + \theta)/2]\sin[(\theta_{k+2} + \theta)/2]}. \tag{2.9}$$

(2.8) is a simple consequence of (2.5).

3. Estimates of $\varphi_k(x)$

Now we prove

LEMMA 3.1. *If* $[(j-1)/n]\pi \leqslant \theta \leqslant (j/n)\pi$ $(j = 1, 2,\ldots, n)$, $x = \cos\theta$, *then we have*

$$|\varphi_k(x)| \leqslant 2^{1/2}, \tag{3.1}$$

$$|\varphi_k(x)| \leqslant \frac{\pi^2}{2(i-1)^3} \quad \text{if } j+1 < k = j+i < n, \text{ or } 1 < k = j-i < j-1, \tag{3.2}$$

$$| \varphi_1(x)| \leqslant \frac{\pi^2}{(i-1)^4} \qquad \text{if } 1 = j - i < j - 1, \tag{3.3}$$

$$| \varphi_n(x)| \leqslant \frac{\pi^2}{(i-1)^4} \qquad \text{if } j + 1 < n = j + i, \tag{3.4}$$

$$| \varphi_k(x) + \varphi_{k+1}(x)| \leqslant \frac{\pi^3}{2(i-1)^4} \quad \text{if } j + 1 < k = j + i < n, \tag{3.5}$$

$$| \varphi_k(x) + \varphi_{k-1}(x)| < \frac{\pi^3}{(i-1)^4} \quad \text{if } 1 < k = j - i < j - 1. \tag{3.6}$$

Proof. Equation (3.1) follows from (1.5) and known estimates of L. Fejer (see [3]).

$$| \ell_k(x)| \leqslant 2^{1/2}. \tag{3.7}$$

Now we prove (3.2). For this purpose, we need

$$\left| \sin \frac{\theta_k - \theta}{2} \right| \geqslant \sin \frac{(2i-1)\pi}{2n} \geqslant \frac{(2i-1)}{2n} \tag{3.8}$$

which follows from $\sin \theta \geqslant (2/\pi)\theta\, (0 < \theta \leqslant \pi/2), (j-1)\pi/n \leqslant \theta \leqslant (j\pi/n)$, $j = 1, 2, \ldots, n$ for the cases $j + 1 < k = j + i < n$ or $1 < k = j - i < j - 1$. Proof of (3.3) and (3.4) follows from (2.6), (2.7), and (3.8). Proof of (3.5) is a simple consequence of (2.8), (2.9), and (3.8). Proof of (3.6) is similar to (3.5), so we omit the details. This proves the lemma.

LEMMA 3.2 (O. Kis [4]). *Let* $(j-1)\pi/n \leqslant \theta \leqslant j\pi/n\; j = 1, 2, \ldots, n$, $x = \cos \theta$. *Then we have*

$$| f(x_k) - f(x)| \leqslant 2\omega \frac{(1-x^2)^{1/2}}{n} + 2\omega \frac{(1)}{n^2}, \; k = j - 1, j, j + 1. \tag{3.9}$$

If $j + 1 < k = j + i < n$ *or* $1 < k = j - i < j - 1$ *then we have*

$$| f(x_k) - f(x)| \leqslant 5\omega \left(\frac{(i(1-x^2)^{1/2})}{n}\right) + 13\omega \left(\frac{i^2}{n^2}\right). \tag{3.10}$$

In the case $j < k = j + i < n$

$$| f(x_k) - f(x_{k+1})| \leqslant 4\omega \left(\frac{(1-x^2)^{1/2}}{n}\right) + 20\omega \left(\frac{i}{n^2}\right), \tag{3.11}$$

and if $1 < k = j - i < j$ *then*

$$| f(x_k) - f(x_{k-1})| \leqslant 4\omega \left(\frac{(1-x^2)^{1/2}}{n}\right) + 20\omega \left(\frac{i}{n^2}\right). \tag{3.12}$$

4. Proof of Theorem

Lemma 3.1 and Lemma 3.2 easily prove our theorem. From (1.5) it follows that

$$\sum_{k=1}^{n} \varphi_k(x) = \sum_{k=1}^{n} \ell_k(x) = 1. \tag{4.1}$$

On using (1.4) and (4.1) we have

$$\begin{aligned} R_n[f, x] - f(x) &= \sum_{k=1}^{n} (f(x_k) - f(x)\, \varphi_k(x) \equiv \sum_{k=1}^{n} u_k(x), \\ &= \sum_{k=1}^{j-2} u_k(x) + u_{j-1}(x) + u_j(x) + u_{j+1}(x) + \sum_{k=j+2}^{n} u_k(x). \end{aligned} \tag{4.2}$$

Of course, if $j = 1$ or 2 (or $n - 1$, n) the first (or last) summation will not appear. By (3.1) and (3.9) it follows that

$$| u_j(x)| = | f(x_j) - f(x)||\, \varphi_j(x)| \leqslant 2\, 2^{1/2} \left[\omega\left(\frac{(1 - x^2)^{1/2}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right]. \tag{4.3}$$

We have similar estimates for $u_{j-1}(x)$ and $u_{j+1}(x)$. Estimates of $\sum_{k=1}^{j-2} u_k(x)$ and $\sum_{k=j+2}^{n} u_k(x)$ are similar. Let us now estimate

$$T \equiv \sum_{k=j+2}^{n} (f(x_k) - f(x)\, \varphi_k(x).$$

For this purpose, we use the idea of grouping the sum in pairs. If the number of terms in the sum is even then we write

$$T = (u_{j+2}(x) + u_{j+3}(x)) + \cdots + (u_{n-1}(x) + u_n(x)).$$

In the other case we have

$$T = (u_{j+2}(x) + u_{j+3}(x)) + \cdots + (u_{n-2}(x) + u_{n-1}(x)) + u_n(x).$$

On using Lemma 3.1 and Lemma 3.2 we obtain

$$\begin{aligned} &|(f(x_k) - f(x))\, \varphi_k(x) + (f(x_{k+1}) - f(x)\, \varphi_{k+1}(x)| \\ &\quad \equiv |(f(x_k) - f(x))(\varphi_k(x) + \varphi_{k+1}(x)) + (f(x_{k+1}) - f(x_k))\, \varphi_{k+1}(x)|, \\ &\quad \leqslant \frac{16\pi^3}{(i-1)^2} \left[\omega\left(\frac{(1 - x^2)^{1/2}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right]. \end{aligned}$$

Hence we have

$$| T | \leqslant 16\pi^3 \left[\omega\left(\frac{(1-x^2)^{1/2}}{n}\right) + \omega\left(\frac{1}{n^2}\right)\right]\left(\sum \frac{1}{(i-1)^2}\right). \tag{4.4}$$

Here the summation on the right-hand side sums through the values 2, 4, 6,... . Combining (4.2), (4.3), and (4.4) proves the theorem.

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